PERGAMON

International Journal of Solids and Structures 38 (2001) 1089–1103

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Gyrocontinua

Maurizio Brocato [a,*], Gianfranco Capriz [b]

[a] *IEI-CNR, Areadella Ricerca di Pisa, via Alfieri, 1 I-56010 Gheacano-Pisa, Italy*
[b] *Dipartimento di Matematica, via F. Buonarroti, 5, I-56126 Pisa, Italy*

Received 12 May 1999

Abstract

We consider a body endowed with the fine structure of very many, very small gyroscopes and describe it as a sort of composite (perhaps constrained) Cosserat continuum (gyrocontinuum). The gyroscopes are supposed to interact significantly with the macroscopic motion of the body via the torques generated by changes of moment of momentum imposed upon them by the macromotion. Thus, we focus essentially on the consequences of constraining the spins to have fixed intensity about some material axis. We conclude the paper studying how the behaviour of a vibrating beam is affected by the special microstructure. A linear discrete gyroelastic body was already considered; (D' Eleuterio, G.M.T., 1984. J. Appl. Mech. 55, 488–489; D' Eleuterio, G.M.T., Hughes, P.C., 1984. J. Appl. Mech. 51, 415–422); we generalise, expand and put their ideas on a firm basis. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Cosserat continua; Microstructure; Distributed controls; Gyricity

1. Introduction

We conceive a continuum with gyroscopic microstructure as the model of a fabric bearing a large number of very small gyroscopes, spinning about axes that are pin-fixed to a flexible frame: obviously, the movement of the frame will be influenced by the gyroscopes via the reactions arising in the pins.

With a more complex mechanical arrangement, we could further imagine the gyroscopes bound to the flexible frame with pliant gimbals; then, the reactions would depend on the changes of direction of the gyroscopic axes relative to the frame; in this case the full angular velocity of the gyroscopes (and not only its gyroscopic component) would be an independent variable.

Thus, we can envisage the following three different cases:

1. gyroscopes with gyroscopic axes bound via pliant (and possibly controllable) devices to the frame;
2. gyroscopes with 'material' gyroscopic axes and variable (e.g. externally controllable) spin intensity (*Constraint 1*);
3. gyroscopes as above, but with fixed spin intensity (*Constraint 2*).

* Corresponding author. Fax: +390-50-554-342.
E-mail address: brocato@iei.pi.cnr.it (M. Brocato).

PII: S0020-7683(00)00075-5

We begin by studying the first case as the most general one and then discuss the other cases as particular occurrences.

2. The material element

2.1. Geometry and mass

As in Capriz (1989), every material element of our continuum is akin to a dynamic system, the state of which can be described by a finite number of order parameters; here such microscopic system is a rigid gyroscope within an affinely deformable capsule. The capsule participates in the gross deformation of the body. On the contrary, the gyroscopes are rigid; their spin inertia is prominent; the centres of gravity of capsule and gyroscope are assumed to coincide always. Thus, we are concerned with a very peculiar and complex type of Cosserat body.

In a reference natural placement, the element has a volume $\mathrm{d}(\mathrm{vol}_*)$ and its mass is the sum of the mass $\rho^{\mathrm{g}}_*\mathrm{d}(\mathrm{vol}_*)$ of the gyroscope and the mass $\rho^{\mathrm{c}}_*\mathrm{d}(\mathrm{vol}_*)$ of the capsule. Notice that ρ^{g}_* and ρ^{c}_* are not real densities, rather they are virtual densities expressing mass per total (rather than fractional) volume. In any other placement the volume of the element becomes $\mathrm{d}(\mathrm{vol}) = \imath\mathrm{d}(\mathrm{vol}_*)$ (here $\imath$ is the jacobian determinant), whereas masses are separately preserved. Introducing appropriate current virtual densities, conservation of mass is assured if

$$\rho^{\mathrm{g}}\imath = \rho^{\mathrm{g}}_*, \quad \rho^{\mathrm{c}}\imath = \rho^{\mathrm{c}}_*.$$

The gyroscope being rigid, its real density is constant; hence the value v of its current volume fraction changes from its reference value v_* as follows:

$$v = \frac{v_*}{\imath}.$$

On the contrary, the real density of the capsule depends in a more complex manner on $\imath$:

$$\bar{\rho}^{\mathrm{c}} = \frac{1 - v_*}{\imath - v_*}\bar{\rho}^{\mathrm{c}}_*;$$

note that $\imath$ must always exceed v_*, in view of the rigidity of the gyroscope. Use of virtual, rather than real, densities makes some developments simpler.

Always in a reference placement, the Euler tensor of the gyroscopes per unit total mass of the material element will be denoted by E^{g}_*, the corresponding inertia tensor by J^{g}_*:

$$J^{\mathrm{g}}_* := (\mathrm{tr}E^{\mathrm{g}}_*)I - E^{\mathrm{g}}_*$$

(I, the identity tensor); the largest eigenvalue of J^{g}_* by α_1; the other (double) eigenvalue by α_2; the unit vector of the gyroscopic axis by g_*

$$J^{\mathrm{g}}_* = \alpha_1 g_* \otimes g_* + \alpha_2(I - g_* \otimes g_*). \tag{1}$$

In any other placement, the eigenvectors of the inertia tensor will change under a rotation Q (a proper orthogonal tensor) as follows:

$$g = Qg_*, \quad J^{\mathrm{g}} = QJ^{\mathrm{g}}_*Q^{\mathrm{T}}. \tag{2}$$

Below Q will denote the absolute rotation of the gyroscope and $w^{\mathrm{g}} := -1/2\mathbf{e}\dot{Q}Q^{\mathrm{T}}$ ($\mathbf{e}$ is Ricci's tensor), its absolute spin vector.

As the capsule is part of the continuum, its affine deformation is described by the usual tensors: the gradient $F := \partial x/\partial x_*$ or its associated rotation R and stretch U.

If E^c_* is the Euler tensor of the capsule per unit total mass in the reference placement, in the actual placement it is,

$$E^c = FE^c_*F^T.$$

2.2. Kinetic energy

Let us now address the problem of the appropriate expressions for the kinetic energy and of the torque of inertia for each material element.

If $v(x)$ is the gross velocity field of the continuum, let $L := \operatorname{grad} v$, $w^c := -1/2 \operatorname{rot} v$ and $D := \operatorname{sym} L$; w^c, spin vector and D, rate of deformation in the gross motion.

The material element is a compound of capsule and gyroscope; translational terms apart, the kinetic energy of the former component derives from its affine deformation as described by the local value of F, that of the latter is purely rotational and depends on w^g. Thus, the non-translational kinetic energy per unit mass of the material element is given by

$$\kappa := \tfrac{1}{2} w^g \cdot J^g w^g + \tfrac{1}{2}(L^T L) \cdot E^c. \tag{3}$$

In terms of the eigenvectors of J^g (see Eq. (1)), the angular momentum of the gyroscope per unit mass of the element can be written as: ($w^g_{\|g} := (w^g \cdot g)g$ and $w^g_{\perp g} := g \times w^g \times g$),

$$J^g w = \alpha_1 w^g_{\|g} + \alpha_2 w^g_{\perp g}, \tag{4}$$

and its kinetic energy of rotation per unit mass:

$$\tfrac{1}{2} w^g \cdot J^g w^g = \tfrac{1}{2}\alpha_1 {w^g_{\|g}}^2 + \tfrac{1}{2}\alpha_2 {w^g_{\perp g}}^2.$$

The time derivative of κ must be computed taking into account that, whereas the eigenvalues α_1 and α_2 are time independent (even though they could vary from point to point in the continuum we take them as absolute constants here: the continuum is supposed to be 'gyroscopically homogeneous'), the directors g change in time:

$$\rho\dot{\kappa} = \alpha_1 w^g_{\|g} \cdot (w^g_{\|g})^{\cdot} + \alpha_2 w^g_{\perp g} \cdot (w^g_{\perp g})^{\cdot} + E^c \cdot L^T(\dot{L} + L^2).$$

2.3. Kinematics of the gyroscope

The gyroscope within each element is entrained by the capsule; totally as concerns translation, not at all or only partially as concerns rotation.

To describe the motion of entrainment we must stipulate how the linkage between capsule and gyroscope gimbals is effected; the linkage is a matter of engineering design rather than theory. The anchoring could correspond possibly to a situation where the entrainment parameter is the rotation tensor R associated with F, but we believe such a possibility rather abstruse. More practical, we believe, is the following arrangement: take – in the reference configuration – the direction g_*, a particular plane through g_* (decided upon at the design stage), of which the normal is the unit vector a_*, and the normal to g_* within this plane.

As this triad need not be one of principal directions for the stretch U, it will not remain orthogonal; actually we need to involve the orthogonal triad, which is determined by the gross motion from the referential one according to the following rules:

$$\hat{g} := \frac{Fg_*}{|Fg_*|}, \quad \hat{a} := \frac{F^{-T}a_*}{|F^{-T}a_*|}, \quad \hat{a} \times \hat{g}. \tag{5}$$

The entrainment rotation of the capsule $\hat{Q}$, i.e. the orthogonal tensor which moves g_* into $\hat{g}$ and the plane normal to a_* into the plane normal to $\hat{a}$, is completely determined by the gross motion (see Eq. (5)),

$$\hat{Q} = \frac{Fg_*}{|Fg_*|} \otimes g_* + \frac{F^{-\mathrm{T}}a_*}{|F^{-\mathrm{T}}a_*|} \otimes a_* + \left(\frac{F^{-\mathrm{T}}a_*}{|F^{-\mathrm{T}}a_*|} \times \frac{Fg_*}{|Fg_*|} \right) \otimes (a_* \times g_*). \tag{6}$$

Having at our disposal the essential notation we are now able to make explicit conditions 3, 1, 2 mentioned in the introduction. In case 1, the gross motion (more precisely the local value of F) does not impose constraints on Q; however, we presume that there be an interaction due, say, to an elastic bond opposing any discrepancy of $\hat{Q}$ from Q or to a control over such discrepancy, then, we need to split Q into the product $G\hat{Q}$, where G, an objective orthogonal tensor, is the rotation of the gyroscope relative to the deformed capsule; i.e. a rotation about $\hat{g}$ in cases 2 and 3, but arbitrary, otherwise:

$$g = G\hat{g}. \tag{7}$$

In case 1, the elastic bond imagined above depends on the whole of G; alternatively the whole G could be one of the control variables. In any case, G describes the relative motion of the gyroscope. The relative motion could be constrained, e.g. the speed of relative rotation could be fixed (case 3) or controlled (case 2); in cases 2 and 3, the relative precession would be constrained to vanish. The choice, in case 2, is to clinch the gyroscopic axis to the capsule and operate on the gyroscopic velocity as the only (scalar) control variable, in which case, $\hat{g}$ would be an eigenvector of G and the relative spin of the gyroscope w^r would be parallel to $g = \hat{g}$:

$$G = \mathrm{e}^{-\theta \mathbf{e}\hat{g}}, \quad w^{\mathrm{r}} = \dot{\theta}\hat{g}$$

with θ, angle of rotation of the gyroscope about its axis $\hat{g}$, counted from the plane of normal $\hat{a}$.

In general, Eqs. (5) and (7) entail for the evolution of any vector attached to the gyroscope as, for instance, g the rule,

$$\dot{g} = [\dot{G}G^{\mathrm{T}} + G\dot{\hat{Q}}\hat{Q}^{\mathrm{T}}G^{\mathrm{T}}]g \tag{8}$$

calling w^{r} and w^{e}, the respective relative and entrainment spin vectors of the gyroscope,

$$w^{\mathrm{r}} := -\tfrac{1}{2}\mathbf{e}\dot{G}G^{\mathrm{T}}, \quad w^{\mathrm{e}} := -\tfrac{1}{2}\mathbf{e}\dot{\hat{Q}}\hat{Q}^{\mathrm{T}}, \tag{9}$$

then, Eq. (8) becomes

$$\dot{g} = w^{\mathrm{g}} \times g, \quad w^{\mathrm{g}} = w^{\mathrm{r}} + Gw^{\mathrm{e}}. \tag{10}$$

Recalling Eq. (6) we can express w^{e} making explicit its dependence on D and w^{c}. Let us first introduce the symbol, w_n^D, for the spin of a unit vector n under a deformation rate D:

$$w_n^D := n \times Dn,$$

and the symbol P_n for the projection of vectors in the plane normal to n:

$$P_n = I - n \otimes n.$$

Then, we have

$$w^{\mathrm{e}} = w^{\mathrm{c}} + \tfrac{1}{2}(I + P_{\hat{a}\times\hat{g}})(w_{\hat{g}}^D - w_{\hat{a}}^D)$$

or, introducing the third order tensor $\mathbf{h} : \mathscr{V} \to \mathrm{Sym}$,

$$\mathbf{h} := -\hat{g} \otimes \mathbf{e}\hat{g} - [(\mathbf{e}\hat{g}) \otimes \hat{g}]^{\mathrm{T}} + \hat{a} \otimes \mathbf{e}\hat{a} + [(\mathbf{e}\hat{a}) \otimes \hat{a}]^{\mathrm{T}} - 2\,\mathrm{sym}(\hat{a} \otimes \hat{g}) \otimes (\hat{a} \times \hat{g})$$

(where $[a_{ijh}]^{\mathrm{T}} = [a_{hji}]$ and it can be shown that $[\mathbf{e}\hat{g} \otimes \hat{g}]_{ihj} = [[\mathbf{e}\hat{g} \otimes \hat{g}]_{ijh}]^{\mathrm{T}}$), also

$$w^{\rm e} = w^{\rm c} - \tfrac{1}{2}\mathbf{h}^{\rm T} D = -\tfrac{1}{2}(\mathbf{e} + \mathbf{h}^{\rm T})L.$$

In any case, it is convenient to split $w^{\rm r}$ into the sum of its vector components along g and normal to it:

$$w^{\rm r} = \omega g + \gamma g \times p, \tag{11}$$

where ω is the gyroscopic speed and γp (p, an appropriate unit vector) the speed of precession (both relative to the capsule and possibly controlled). Thus, the vector components of the total spin w parallel and normal to g (and thus along principal directions of $J^{\rm g}$) are

$$\begin{aligned} w^{\rm g}_{\|g} &= g(\omega + \omega^{\rm e}), \qquad \omega^{\rm e} := g \cdot G(w^{\rm c} - w^{D}_{\hat{a}}), \\ w^{\rm g}_{\perp g} &= g \times (\gamma p + q^{\rm e}), \qquad q^{\rm e} := -g \times G(w^{\rm c} + w^{D}_{\hat{g}}). \end{aligned} \tag{12}$$

2.4. Remarks

To make some of the previous assumptions clearer, here we explore the consequences of conceiving a different kind of linkage. Let us assume the latter to be a compound of a rigid spherical shell of absolutely negligible mass, included in the capsule and housing the gyroscope. There will be a discrepancy between the simply rotational motion of the inclusion and the affine deformation of the capsule; e.g. we may assume them to be joined via elastic bonds (and take the reference placement as one showing no stresses) with the outer surface of the shell coinciding with the inner surface of the capsule.

Thus, if m_* is a material vector issuing from the center of the shell and initially pointing on its surface, it will become, in the present placement, $m = Fm_*$ while the point on the shell identified by it has moved to $\hat{m} = \hat{Q}m_*$. As for the hypothesis of elasticity, it can be made explicit, at least in quasi-static instances, through a requirement of minimum for a potential depending on the length $|m - \hat{m}|$ of the bond; e.g. it implies for $\hat{Q}$ the condition,

$$\min_{\hat{Q}\in{\rm Orth}^+} m_* \cdot (F^{\rm T} - \hat{Q}^{\rm T})(F - \hat{Q})\, m_* \quad \forall m_*,$$

and thus, $\hat{Q} = R$ by Grioli's theorem (Truesdell and Toupin, 1960).

This result, occurring in a very special case, though interesting for applications, involves constitutive assumptions (made explicit here through the minimum condition in a particular contest) and thus cannot be taken a priori at this stage of the kinematic modelling. Exploring in depth the consequences of the particular construction envisaged here would lead to a more complex theory than the one we aim at establishing, because two microstructures ought to be described: the gyroscope and the shell-linkage.

An ancillary premise to the developments of the following sections is the study of $\hat{Q}$ when the stretch is small; it is possible to show that $\hat{Q}$ differs from R significantly even in this limit case. Let $U = I + E$ and $o(|E|)$ denote any function of E of the order larger than one, thus negligible for small deformations of E when compared with first order terms. Then, by Eq. (5), the rotation $\hat{Q}$ (cf. Eq. (6)) can be written

$$\begin{aligned} \hat{Q} &= R\left[\frac{Ug_*}{|Ug_*|} \otimes g_* + \frac{U^{-1}a_*}{|U^{-1}a_*|} \otimes a_* + \left(\frac{U^{-1}a_*}{|U^{-1}a_*|} \times \frac{Ug_*}{|Ug_*|}\right) \otimes (a_* \times g_*)\right] \\ &= R\big\{I + P_{g^*}Eg_* \otimes g_* - P_{a^*}Ea_* \otimes a_* - [(P_{g^*}Eg_*) \times a_* + (P_{a^*}Ea_*) \times g_*] \otimes a_* \times g_* + o(E)\big\}, \end{aligned}$$

which shows how the difference between $\hat{Q}$ and R depends on first order terms.

2.5. Additional assumptions

Here, we analyse some results of the previous paragraphs and introduce additional assumptions aiming at a reasonable simplification of the problem. The assumptions listed here allow us to compare our model with the results of D'Eleuterio (1984) and D'Eleuterio and Hughes (1984). Furthermore, it will be useful to call upon them when focusing on practical issues.

(1) We disregard the kinetic energy and inertia torque due to the affine motion of the capsule (Assumption 1). This assumption is justified by the idea that, in most concrete instances, the kinetic energy of rotation of the gyroscope, as implied by a very high gyroscopic velocity, will be predominant in the material element.

Thus, developments in the following paragraphs will apply only when the following approximate expression is acceptable (cf. Eq. (3)):

$$\kappa \approx \tfrac{1}{2}\alpha_1[\omega + \omega^{\mathrm{e}}]^2 + \tfrac{1}{2}\alpha_2[g \times (\gamma p + q^{\mathrm{e}})]^2. \tag{13}$$

(2) The first eigenvector of J^{g}, namely α_1, is taken to be much larger than the second, α_2; thus, we may neglect all contributions of the inertia torque of the gyroscope which are the time derivatives of the component of the angular momentum orthogonal to the gyroscopic direction (Assumption 2):

$$\alpha_2 \ll \alpha_1.$$

(3) The component of the total spin along the gyroscopic axis, within the approximation, is due only to the gyroscopic speed, which will be much higher than the component of the spin of entrainment along that axis; under such circumstances, we neglect ω^{e} with respect to ω (Assumption 3):

$$\omega^{\mathrm{e}} \ll \omega.$$

(4) The spin of the linkage on the capsule may be simply (or at least predominantly) given by w^{c} (Assumption 4; e.g. the linkage's triad lays along the principal directions of D):

$$-\tfrac{1}{2}\mathbf{h}^{\mathrm{T}} D \ll w^{\mathrm{c}}.$$

(5) Finally, often the rate of variation of this spin, $\dot{w}^{\mathrm{c}}$, happens to be orthogonal to g (Assumption 5; in quasi-static instances this statement can be taken a priori, deciding upon the module of $\dot{w}^{\mathrm{c}}$):

$$\dot{w}^{\mathrm{c}} \cdot g \ll \omega |w^{\mathrm{c}} \times g|.$$

These five assumptions give the background of the model proposed by D'Eleuterio (1984) and D'Eleuterio and Hughes (1984).

In the following part of the text, we will drop the suffix g from quantities defined on the gyroscope, thus in particular J^{g} will be denoted simply as J, w^{g} as w.

3. Inertia

3.1. The kinetic energy theorem; the inertia of embedded gyroscopes

Eqs. (1) and (2) imply Poisson's formula,

$$\dot{J} = WJ - JW,$$

which, in turn, implies,

$$\dot{J}w = -w \times Jw, \quad w \cdot \dot{J}w = 0,$$

so that

$$\dot{\kappa} = w \cdot J\dot{w} = (Jw)^{\cdot} \cdot w.$$

As per elementary dynamics, we take here for the inertia torque per unit mass,

$$b_{Ig} := (Jw)^{\cdot}, \tag{14}$$

where (cf. Eqs. (4) and (12))

$$(Jw)^{\cdot} = \alpha_1(\omega + \omega^{e})[\gamma p + q^{e}] + \alpha_1 g(\omega + \omega^{e})^{\cdot} + \alpha_2 g \times (\gamma p + q^{e})^{\cdot}. \tag{15}$$

If Constraint 2 applies (i.e. $\gamma = 0$, $G^{T}g = g$ and $g \times Gn = g \times n\ \forall n$ and $\partial\omega/\partial\tau = 0$) Eq. (15) simplifies as follows:

$$(Jw)^{\cdot} = \alpha_1\left[(\omega + g\cdot(w^{c} - w^{D}_{\hat{g}}))(w^{c} + w^{D}_{\hat{a}}) \times g + g((w^{c} - w^{D}_{\hat{g}})^{\cdot} + w^{c} \times w^{D}_{\hat{a}})\cdot g\right] + \alpha_2 P_g (w^{c} - w^{D}_{\hat{a}})^{\cdot}. \tag{16}$$

By Assumption 2, the last term can be neglected; adding Assumption 3, we get the approximate expression:

$$(Jw)^{\cdot} \approx \alpha_1\left[\omega(w^{c} + w^{D}_{\hat{a}}) \times g + g((w^{c} - w^{D}_{\hat{g}})^{\cdot} + w^{c} \times w^{D}_{\hat{a}})\cdot g\right]. \tag{17}$$

Finally, when also Assumptions 4 and 5 hold, we get from Eq. (17), the expression of the inertia torque adopted by D'Eleuterio (1984) and D'Eleuterio and Hughes (1984).

$$(Jw)^{\cdot} \approx \tfrac{1}{2}\alpha_1 \omega g \times \operatorname{rot} v. \tag{18}$$

3.2. Inertia of composite material points

We recall two fundamental properties of inertia relevant, when extending concepts in elementary dynamics to the dynamics of complex continua:

1. the power per unit mass of the inertia forces is the opposite of the time derivative of the kinetic energy per unit mass,
2. the virtual power of inertial forces coincides with the first variation of an appropriate functional.

Condition 1 is satisfied, when the inertia force on the gyroscopes is chosen in accordance with Eq. (14) and that on the capsule is $\rho\dot{v}$:

$$\left(\tfrac{1}{2}v^2 + \tfrac{1}{2}w\cdot Jw\right)^{\cdot} = \dot{v}\cdot v + (Jw)^{\cdot}\cdot w$$

with w depending on $\operatorname{grad} v$ in a manner that needs attention when focusing on condition 2.

The virtual power of accelerations is given by

$$\mathscr{P}^{*}_{\mathrm{acc}} = \int_{V^*} \rho\,(\dot{v}\cdot\hat{v} + b_{Ig}\cdot\hat{w})\,\mathrm{d}V, \tag{19}$$

where the virtual absolute spin of the gyroscopes $\hat{w}$ is equal to:

$$\begin{aligned}\hat{w} &= \hat{w}^{r} + G\hat{w}^{e}\\ &= \hat{\omega}g + g \times \hat{q} - \tfrac{1}{2}G\operatorname{rot}\hat{v} - \tfrac{1}{2}G\mathbf{h}^{T}\operatorname{symgrad}\hat{v}.\end{aligned} \tag{20}$$

Note that the field g of present directions of the gyroscopic axes and the field G of present gyroscopic rotation derives from the time integration of the spin equation ($\dot{g} = w \times g$ and $\dot{G} = -(\mathbf{e}w)G$). Arbitrary variations are allowed only for the position of the capsule (and they are expressed through the virtual velocity $\hat{v}$), the angular coordinate of the gyroscope relative to the capsule (the virtual angular velocity $\hat{\omega}$) and the inclination of the gyroscope within it (the virtual precession velocity $\hat{q}$).

By Eq. (20),

$$b_{Ig}\cdot\hat{w} = \beta_I\hat{\omega} + \bar{b}_{Iq}\cdot\hat{q} + \tfrac{1}{2}\operatorname{div}[(\mathbf{e}+\mathbf{h})(G^{T}b_{Ig})]\cdot\hat{v} - \tfrac{1}{2}\operatorname{div}\{[(\mathbf{e}+\mathbf{h})(G^{T}b_{Ig})]^{T}\hat{v}\}, \tag{21}$$

where the symbols

$$\beta_I := b_{Ig} \cdot g, \qquad \bar{b}_{Iq} := b_{Ig} \times g,$$

were introduced. By integration over a fit region V^*, we get

$$\mathscr{P}^*_{\rm acc} = \int_{V^*} \rho\{[\dot{v} + \tfrac{1}{2}{\rm div}[(\mathbf{e}+\mathbf{h})(G^{\rm T} b_{Ig})]] \cdot \hat{v} + \beta_I \hat{\omega} + \bar{b}_{Iq} \cdot \hat{q}\}\,{\rm d}V - \tfrac{1}{2}\int_{\partial V^*} \rho\hat{v} \cdot [(\mathbf{e}+\mathbf{h})(G^{\rm T} b_{Ig})]n\,{\rm d}S. \tag{22}$$

We may interpret this formula as indicating that the gyroscopic rotational inertia contributes to translational inertia in the bulk and on the boundary.

Within the hypotheses that led to Eq. (18), the above mentioned contributions simplify; in particular, the body force per unit mass due to the torque of inertia of the gyroscopes becomes

$$\tfrac{1}{2}{\rm div}[(\mathbf{e}+\mathbf{h})(G^{\rm T} b_{Ig})] \approx -\tfrac{1}{2}{\rm rot}(\tfrac{1}{2}\alpha_1 \omega g \times {\rm rot}\, v).$$

4. Balance laws of the continuum

In the previous paragraph, the state of the material element was described through some fixed parameters (the eigenvalues α_1 and α_2 of $J^{\rm g}$) and a variable vector giving axis and amount of a rotation from a reference state: the typical microstructure of a Cosserat continuum; however, formal aspects apart, the fabric of the device indwelling each material element lends specific properties to our body. To linger, for the moment, on formal aspects, advantage can be taken of results available in the literature (Capriz, 1989; Capriz and Giovine, 1997).

Conservation of mass and balance of momentum are expressed, as usual, through the equation of continuity and Cauchy's equation; though the stress tensor T need not be symmetric: the equation of balance of moment of momentum requires that

$$\mathbf{e}T = \mathscr{A}^{\rm T}\zeta + ({\rm grad}\,\mathscr{A}^{\rm T})\mathscr{S}. \tag{23}$$

A last balance equation steps in for microstructural quantities:

$$\rho\left(\frac{\partial\chi}{\partial\dot{v}} - \frac{\partial\chi}{\partial v}\right) = \rho\beta - \zeta + {\rm div}\,\mathscr{S}.$$

Here $\mathscr{A}$ denotes the infinitesimal generator of rotations for the microstructure, $-\zeta$ the equilibrated microforce and $\mathscr{S}$ the microstress tensor, χ is the density of the kinetic co-energy of the microstructure v, β the density per unit mass of the external actions on the microstructure.

All these terms may be displayed into a more precise form once the microstructure is defined. In our case, it is convenient to express the absolute rotation of the gyroscope Q through its axial vector q:

$$Q = {\rm e}^{-\mathbf{e}q},$$

so that we may let v be q; then $\dot{v}$ is $w := -\frac{1}{2}\mathbf{e}\dot{Q}Q^{\rm T} = \dot{q}$.

If we describe also the rotation of the observer through an axial vector r, $\mathscr{A}$ becomes a second order tensor A (cf. also Eq. (10): in a rigid rotation r of the gyrocontinuum with gyroscopes at rest with respect to their capsules, their spin is $\dot{q} = G\dot{r}$):

$$A = \left.\frac{\partial q(r)}{\partial r}\right|_{r=0} = G.$$

Therefore, Eq. (23) takes the explicit form

$$\mathbf{e}T = G^{\rm T}z + ({\rm grad}\,G^{\rm T})S, \tag{24}$$

where z and S are the vector (torque per unit volume) and second order tensor (torque per unit oriented surface) representing ζ and $\mathscr{S}$ in the present model.

Recalling results displayed in Section 3.1, the balance of micromomentum is ruled by the equations:

$$\begin{cases} -z + \mathrm{div} S + \rho b_g = \rho b_{Ig}, & x \in V_\tau, \\ Sn = f_g, & x \in \partial V_\tau, \end{cases} \tag{25}$$

where b_g and f_g are external torques applied in the unit mass and on the unit surface respectively, and V_τ and ∂V_τ are the domain occupied by the body in the present configuration and its surface respectively.

The equations of balance of momentum for the macromotion, by the result of Section 3.2, become

$$\begin{cases} \mathrm{div}\, T + b = \rho \dot{v} + \frac{1}{2}\rho\, \mathrm{div}\, [(\mathbf{e} + \mathbf{h})(G^{\mathrm{T}} b_{Ig})], & x \in V_\tau, \\ Tn = f + \frac{1}{2}\rho[(\mathbf{e} + \mathbf{h})(G^{\mathrm{T}} b_{Ig})]n, & x \in \partial V_\tau. \end{cases} \tag{26}$$

4.1. Unconstrained gyroscopes' spins – splitting of micromomentum

The power of z and S can be decomposed so as to put in evidence the contributions due to the movement of the gyroscope relative to the capsule:

$$z \cdot w^{\mathrm{r}} + S \cdot \mathrm{grad}\, w^{\mathrm{r}} = \zeta\omega + s \cdot \mathrm{grad}\, \omega + \bar{z} \cdot q + \bar{S} \cdot \mathrm{grad}\, q \tag{27}$$

having defined the fields:

$$\begin{aligned} &\zeta := z \cdot g + S \cdot \mathrm{grad}\, g, \\ &s := S^T g, \\ &\bar{z} := z \times g - \mathbf{e}[(\mathrm{grad}\, g)S^{\mathrm{T}}], \\ &\bar{S} := (\mathbf{e}g)S; \end{aligned} \tag{28}$$

their constitutive features seem easier to understand than those of z and S directly: ζ is the intensity of a couple per unit volume acting along the gyroscopic axis, $\bar{z}$ is a torque per unit volume working with changes of directions of that axis, $s \cdot n$ and $\bar{S}n$ are the corresponding objects per unit oriented surface.

Note the inverse relations:

$$\begin{aligned} &z = \zeta g + g \times \bar{z} - (S \cdot \mathrm{grad}\, g)g + 2\mathrm{skw}[(\mathrm{grad}\, g)S]g, \\ &S = s \otimes g - (\mathbf{e}g)\bar{S}, \end{aligned}$$

and, with the use of Eq. (28):

$$\begin{aligned} &-\zeta + \mathrm{div}\, s = (-z + \mathrm{div}\, S) \cdot g, \\ &-\bar{z} + \mathrm{div}\, \bar{S} = (-z + \mathrm{div}\, S) \times g, \\ &s \cdot n = g \cdot Sn, \qquad \bar{S}n = (Sn) \times g. \end{aligned}$$

From Eqs. (24)–(26), we get the local equations of balance of micromomentum in terms of components along g and orthogonal to g:

$$\begin{cases} \left.\begin{aligned} -\zeta + \mathrm{div}\, s + \rho\beta = \rho\beta_I \\ -\bar{z} + \mathrm{div}\, \bar{S} + \rho\bar{b}_q = \rho\bar{b}_{Iq} \end{aligned}\right\} & x \in V_\tau, \\ \left.\begin{aligned} s \cdot n = \phi \\ \bar{S}n = \bar{f}_q \end{aligned}\right\} & x \in \partial V_\tau, \end{cases} \tag{29}$$

where the components of the external actions on gyroscopes are:

$$\beta := b_g \cdot g, \quad \bar{b}_q := b_g \times g, \quad \phi := f_g \cdot g, \quad \bar{f}_q := f_g \times g.$$

4.2. Constrained gyroscopes – variable spins about material axis

If Constraint 1 applies (each gyroscope turns about a material axis), we have to add to the analysis developed in the previous section, the kinematic constraint of null relative precession:

$$\gamma = 0 \iff w = \omega g - \tfrac{1}{2}G(\mathbf{e} + \mathbf{h}^{\mathrm{T}})\mathrm{grad}\, v. \tag{30}$$

To determine the reactions to this constraint, we assume it to be perfect: microforces and stresses can be decomposed into active and reactive parts, and the density of the internal reactive power is null for any motion allowed by the constraint.

The virtual absolute spin of the gyroscopes is given by Eq. (20); in the present, constrained case its gradient is equal to

$$\mathrm{grad}\, \hat{w} = \hat{\omega}\, \mathrm{grad}\, g + g \otimes \mathrm{grad}\, \hat{\omega} - \tfrac{1}{2}\mathrm{grad}(G\,\mathrm{rot}\, \hat{v}) - \tfrac{1}{2}\mathrm{grad}(G\mathbf{h}^{\mathrm{T}}\mathrm{symgrad}\, \hat{v}). \tag{31}$$

Consequently, the density of the internal power of the reactive stresses becomes

$$\begin{aligned}\overset{r}{\pi}_{\mathrm{int}} := {} & \overset{r}{\zeta}\hat{\omega} + \overset{r}{s} \cdot \mathrm{grad}\, \hat{\omega} - \tfrac{1}{2}\{\mathbf{h}(G^T\overset{r}{z}) + \mathbf{h}[(\mathrm{grad}\, G^T)\overset{r}{S}] - 2\mathrm{sym}\, \overset{r}{T}\} \\ & \cdot \mathrm{symgrad}\, \hat{v} - \tfrac{1}{2}[G^T\overset{r}{z} + (\mathrm{grad}\, G^T)\overset{r}{S} - \mathbf{e}\overset{r}{T}] \cdot \mathrm{rot}\, \hat{v} - \tfrac{1}{2}(G^T\overset{r}{S}) \cdot \mathrm{grad}[(\mathbf{e} + \mathbf{h}^T)\mathrm{grad}\, \hat{v}].\end{aligned} \tag{32}$$

Thus, condition $\overset{r}{\pi}_{\mathrm{int}} = 0$, $\forall \hat{v}$, $\forall \hat{\omega}$ implies

$$\begin{aligned}& \overset{r}{\zeta} = 0, \quad \overset{r}{s} = 0, \\ & \mathbf{h}(G^T\overset{r}{z}) + \mathbf{h}[(\mathrm{grad}\, G^T)\overset{r}{S}] - 2\,\mathrm{sym}\, \overset{r}{T} = 0, \\ & G^T\overset{r}{z} + (\mathrm{grad}\, G^T)\overset{r}{S} - \mathbf{e}\overset{r}{T} = 0, \\ & \overset{r}{S} = 0;\end{aligned} \tag{33}$$

and, by definitions (28) and decomposition (11) for $\overset{r}{z}$, we find

$$\overset{r}{\bar{S}} = 0, \qquad \overset{r}{z} = g \times \overset{r}{\bar{z}}, \qquad \overset{r}{T} = \tfrac{1}{2}(\mathbf{e} + \mathbf{h})(G^T\overset{r}{z}). \tag{34}$$

We conclude that the only reaction entering the equilibrium equations is $\overset{r}{\bar{z}}$, which can be evaluated through the second of Eq. (29) and introduced in the Cauchy stress tensor,

$$T = \overset{a}{T} + \tfrac{1}{2}(\mathbf{e} + \mathbf{h})[G^T(g \times \overset{r}{\bar{z}})], \tag{35}$$

before solving Eq. (26). Finally the first of Eq. (29) is ‘pure’ – as no traces of the reactions – appear there.

4.3. Constrained gyroscopes – constant spins about material axis

Let us now add the constraint of fixed spin intensity (Constraint 2):

$$\omega = \bar{\omega} = \mathrm{const.} \tag{36}$$

Then, the terms in the virtual internal power Eq. (32) involving $\hat{\omega}$ vanish. Thus, we have no information about $\overset{r}{\zeta}$, while $\overset{r}{s}$ is still null as a consequence of $\overset{r}{S} = 0$; the other conclusions, second and first of Eq. (33), first of Eq. (34) and (35), drawn in the previous case still hold true, while the first of Eq. (29) contains the reaction $\overset{r}{z}$, which now has a non-zero component along g.

The reaction $\overset{\mathrm{r}}{\bar{z}}$, normal to the gyroscopic axis, can still be evaluated by the second of Eq. (29), while the constraint of constant gyroscopic speed gives for $\overset{\mathrm{r}}{\zeta}$:

$$\overset{\mathrm{r}}{\zeta} = -\overset{a}{\zeta} + \operatorname{div}\overset{a}{s} + \rho\beta - \rho\alpha_1\dot{\omega}^{\mathrm{e}}.$$

As already mentioned, under this constraint the expression (15) of the derivative of the angular momentum simplifies to Eq. (16).

5. Examples

5.1. Plane waves

To offer an example of application, we consider a linear elastic homogeneous material with Lamé's constants λ and μ, assume small perturbations from a natural equilibrium state described through the field of small displacements u, neglect body forces b and take s null. We consider the case of Constraint 2 (constrained gyroscopes spinning about material axes with constant intensity) and accept the approximate expression (17) for the inertia force. Then, the equation of motion reduces to

$$\rho\ddot{u} - \mu\Delta u - (\lambda+\mu)\operatorname{grad}\operatorname{div} u - \tfrac{1}{2}\operatorname{rot}\left[\tfrac{1}{2}k \times \operatorname{rot}\dot{u} + (\operatorname{grad} k)\dot{u}\right] = 0, \tag{37}$$

where $k = \rho\alpha_1\omega g$ can be called the 'gyricity' field (cf. D'Eleuterio (1984) for this name). Assume k to be constant (almost everywhere) and uniform and a reference be chosen with first axis along g.

Let us look for solutions of Eq. (37) depending on one spatial coordinate, x_1, and time, τ:

$$\begin{cases} \rho u_{1,\tau\tau} - (\lambda+2\mu)u_{1,11} = 0, \\ \rho u_{2,\tau\tau} - \mu u_{2,11} - \frac{\rho\alpha_1\omega}{4} u_{3,11\tau} = 0, \\ \rho u_{3,\tau\tau} - \mu u_{3,11} + \frac{\rho\alpha_1\omega}{4} u_{2,11\tau} = 0, \end{cases}$$

under the form ($j \in \{1,2,3\}$):

$$u_j = \exp(\chi_j x_1 + \varsigma\tau) \tag{38}$$

with complex χ_j and ς.

From Eq. (37), we get the three conditions on the four unknown complex numbers χ_j, ς:

$$\begin{cases} \chi_1^2 - \frac{\rho\varsigma^2}{\lambda+2\mu} = 0, \\ \chi_2^2 + \frac{\rho\alpha_1\omega\varsigma}{4\mu}\chi_3^2 - \frac{\rho\varsigma^2}{\mu} = 0, \\ \chi_3^2 - \frac{\rho\alpha_1\omega\varsigma}{4\mu}\chi_2^2 - \frac{\rho\varsigma^2}{\mu} = 0, \end{cases} \tag{39}$$

and a solution in terms of ς:

$$\chi_1 = \varsigma\sqrt{\frac{\rho}{\lambda+2\mu}}, \qquad \chi_2 = \varsigma\sqrt{\frac{\rho}{\mu}}\sqrt{\frac{1-\frac{\rho\alpha_1\omega\varsigma}{4\mu}}{1+\left(\frac{\rho\alpha_1\omega\varsigma}{4\mu}\right)^2}}, \qquad \chi_3 = \varsigma\sqrt{\frac{\rho}{\mu}}\sqrt{\frac{1+\frac{\rho\alpha_1\omega\varsigma}{4\mu}}{1+\left(\frac{\rho\alpha_1\omega\varsigma}{4\mu}\right)^2}}. \tag{40}$$

If $\rho\alpha_1\omega\varsigma/4\mu = \mathrm{i}$ (here $\mathrm{i}^2 = -1$), the second and third expressions of Eq. (39) give $\varsigma = 0$; thus, solution Eq. (40) exists for all real ω. If $\omega \to \infty$, both χ_2 and χ_2 tend to zero.

Computing the square roots in Eq. (40) in case, ς is a pure complex number, one gets (signs are immaterial):

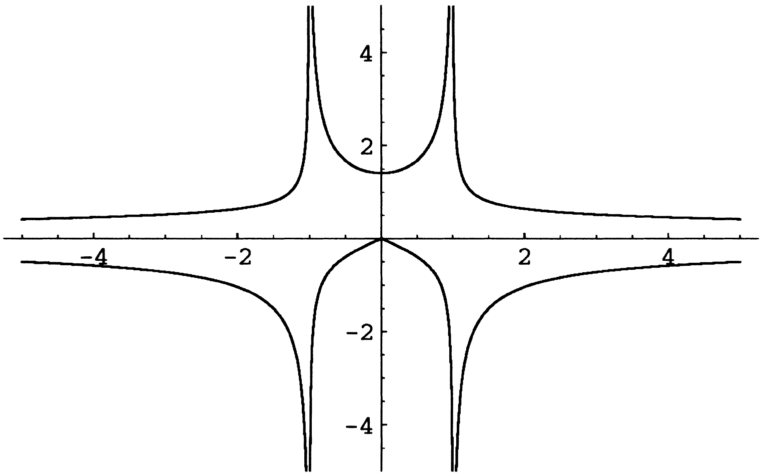

Fig. 1. Plot of $\mathrm{Re}\left(\chi_2/\sigma\sqrt{2\mu/\rho}\right)$ (below) and $\mathrm{Im}\left(\chi_2/\sigma\sqrt{2\mu/\rho}\right)$ (above) versus $\rho\alpha_1\omega\sigma/4\mu$.

$$\chi_2 = \chi_3 = \sigma\sqrt{\frac{\rho}{2\mu}}\frac{\mathrm{i}\sqrt{1+\sqrt{1+\left(\frac{\rho\alpha_1\omega\sigma}{4\mu}\right)^2}} - \sqrt{-1+\sqrt{1+\left(\frac{\rho\alpha_1\omega\sigma}{4\mu}\right)^2}}}{\sqrt{1-\left(\frac{\rho\alpha_1\omega\sigma}{4\mu}\right)^2}}, \tag{41}$$

and therefore $\omega \neq 0 \Rightarrow \mathrm{Re}[\chi_j] \neq 0$. This last remark shows that for $\omega \neq 0$ solutions of the kind (38) are always possible, but with χ_2 and χ_3 never purely imaginary, i.e. an exponential increase or decay of the solution along x_1 is always present.

In Fig. 1, we plot the real and the imaginary parts of solution (41) versus $\rho\alpha_1\omega\sigma/4\mu$; values of the latter variable close to ± 1 give the larger gyroscopic effect on the wavelengths.

5.2. Vibrations of beams

A second example that illustrates the possibilities of a mechanical system with embedded gyroscopes is that of an Euler–Bernoulli linear elastic beam experiencing small displacements u. To adapt our general results to this particular case, we need to introduce the beam's geometry, i.e. a slender cylinder along x_1, with the kinematic constraint of conservation of normal cross-sections; we write the displacement field as follows:

$$\begin{cases} u_1 = \bar{u}_1 - \bar{u}_{3,1}x_3 - \bar{u}_{2,1}x_2, \\ u_2 = \bar{u}_2, \\ u_3 = \bar{u}_3. \end{cases} \qquad \bar{u} = \bar{u}(x_1).$$

Developments are straightforward if one takes the axes x_2 and x_3 with origin in the centre and along the principal inertia axes of the cross-section – calling ρ_2 and ρ_3, their relative radii of inertia, and E, the Young's modulus of the material, the beam is made of – and if one neglects the rotational inertia of cross-sections in writing the equations of motion of the beam. Furthermore we assume the micro-stress tensor S be null and take a simplified form for the inertia forces of the gyroscopes valid if their axes are fixed to the main body and their spin is constant in time and in space (Constraint 2).

Therefore, the two equations of lateral motion of the beam are

$$\begin{cases} \bar{u}_{2,\tau\tau} + \frac{E\rho_3^2}{\rho}\bar{u}_{2,1111} + \frac{1}{2}\alpha_1\omega\bar{u}_{3,11\tau} = 0, \\ \bar{u}_{3,\tau\tau} + \frac{E\rho_2^2}{\rho}\bar{u}_{3,1111} - \frac{1}{2}\alpha_1\omega\bar{u}_{2,11\tau} = 0. \end{cases} \tag{42}$$

If the beam is pinned at the ends $x = 0$ and $x = \bar{x}$, then the wavelengths must be an integer fraction of $\bar{x}/\pi$ and we can start from the assumption:

$$\bar{u}_2 = \exp\left(\pm\,\mathrm{i}\frac{\iota_2\pi x}{\bar{x}} + \frac{\tau}{\bar{\tau}_2} \pm \mathrm{i}\sigma_2\tau\right), \qquad \bar{u}_3 = \exp\left(\pm\,\mathrm{i}\frac{\iota_3\pi x}{\bar{x}} + \frac{\tau}{\bar{\tau}_3} \pm \mathrm{i}\sigma_3\tau\right)$$

(with integers ι_2 and ι_3 and real $\bar{\tau}_2$, $\bar{\tau}_3$, σ_2 and σ_3).

Eqs. (42) lead to the conditions:

$$\begin{cases} \frac{1}{\bar{\tau}_2^2} - \sigma_2^2 + \frac{E\rho_3^2}{\rho}\left(\frac{\iota_2\pi}{\bar{x}}\right)^4 - \frac{1}{2}\alpha_1\omega\left(\frac{\iota_3\pi}{\bar{x}}\right)^2\frac{1}{\bar{\tau}_3} = 0, \\ \frac{1}{\bar{\tau}_3^2} - \sigma_3^2 + \frac{E\rho_2^2}{\rho}\left(\frac{\iota_3\pi}{\bar{x}}\right)^4 + \frac{1}{2}\alpha_1\omega\left(\frac{\iota_2\pi}{\bar{x}}\right)^2\frac{1}{\bar{\tau}_2} = 0, \\ \frac{\sigma_2}{\bar{\tau}_2} = \frac{\alpha_1\omega}{4}\left(\frac{\iota_3\pi}{\bar{x}}\right)^2\sigma_3, \\ \frac{\sigma_3}{\bar{\tau}_3} = -\frac{\alpha_1\omega}{4}\left(\frac{\iota_2\pi}{\bar{x}}\right)^2\sigma_2, \end{cases}$$

and thus,

$$\frac{1}{\bar{\tau}_2} = \frac{\alpha_1\omega}{4}\left(\frac{\iota_3\pi}{\bar{x}}\right)^2\frac{\sigma_3}{\sigma_2}, \qquad \frac{1}{\bar{\tau}_3} = -\frac{\alpha_1\omega}{4}\left(\frac{\iota_2\pi}{\bar{x}}\right)^2\frac{\sigma_2}{\sigma_3},$$

$$\begin{cases} \frac{E\rho_3^2}{\rho}\iota_2^4 + \left(\frac{\alpha_1\omega}{4}\right)^2\left(\frac{\sigma_3}{\sigma_2}\right)^2\iota_3^4 + 2\left(\frac{\alpha_1\omega}{4}\right)^2\frac{\sigma_2}{\sigma_3}\iota_3^2\iota_2^2 = \left(\frac{\bar{x}}{\pi}\right)^4\sigma_2^2, \\ \frac{E\rho_2^2}{\rho}\iota_3^4 + \left(\frac{\alpha_1\omega}{4}\right)^2\left(\frac{\sigma_2}{\sigma_3}\right)^2\iota_2^4 + 2\left(\frac{\alpha_1\omega}{4}\right)^2\frac{\sigma_3}{\sigma_2}\iota_3^2\iota_2^2 = \left(\frac{\bar{x}}{\pi}\right)^4\sigma_3^2. \end{cases} \tag{43}$$

The first two equations give the time constant characterising excitation or damping of vibrations; note that, τ_2 having sign opposite of that of τ_3, if movements are damped in direction x_3, they are excited in the orthogonal direction. If diversion of energy from one mode to another is sought, the system is of increasing proficiency, if $\omega > 0$, when $\sigma_3 \ll \sigma_2$, i.e. the more the beam is stiff against bending around direction x_3 than around x_2, the faster the energy content of movements in direction x_2, is diverted to oscillations in the orthogonal direction (the case $\omega < 0$ leading to symmetric conclusions).

Notice also that the effect of gyroscopes is greater for the higher modes with geometric progression.

The second couple of expressions in Eq. (43) determines the circular frequencies σ_2 and σ_3, for every settled couple of integer numbers (ι_2, ι_3). Synchronous solutions (i.e. solutions with equal frequencies) for the two orthogonal motions exist under condition:

$$\sqrt{\frac{E\rho_3^2}{\rho}} > \frac{\alpha_1\omega}{4} > \sqrt{\frac{E\rho_2^2}{\rho}}$$

or

$$\sqrt{\frac{E\rho_3^2}{\rho}} < \frac{\alpha_1\omega}{4} < \sqrt{\frac{E\rho_2^2}{\rho}},$$

and have wavelengths satisfying the relation:

$$\iota_2\sqrt[4]{\frac{E\rho_3^2}{\rho} - \left(\frac{\alpha_1\omega}{4}\right)^2} = \iota_3\sqrt[4]{-\frac{E\rho_2^2}{\rho} + \left(\frac{\alpha_1\omega}{4}\right)^2},$$

a condition which is worth comparing with that occurring for $\omega = 0$:

$$\frac{\iota_2}{\iota_3} = \sqrt{\frac{\rho_2}{\rho_3}}.$$

Particular solutions are those with equal characteristic time (in absolute value):

$$|\bar{\tau}_2| = |\bar{\tau}_3| = \bar{\tau} \quad \Rightarrow \quad \iota_3\sigma_3 = \pm\iota_2\sigma_2;$$

in this case the two conditions linking the circular frequencies and the indices of wavelengths become (cf. Eq. (43)):

$$\begin{cases} \left[\frac{E\rho_3^2}{\rho} + \left(\frac{\alpha_1\omega}{4}\right)^2\left(\frac{\sigma_2}{\sigma_3}\right)^2\left(1 + 2\frac{\sigma_2}{\sigma_3}\right)\right]\iota_2^4 = \left(\frac{\bar{x}}{\pi}\right)^4\sigma_2^2, \\ \left[\frac{E\rho_2^2}{\rho} + \left(\frac{\alpha_1\omega}{4}\right)^2\left(\frac{\sigma_3}{\sigma_2}\right)^2\left(1 + 2\frac{\sigma_3}{\sigma_2}\right)\right]\iota_3^4 = \left(\frac{\bar{x}}{\pi}\right)^4\sigma_3^2. \end{cases} \tag{44}$$

In the case $\rho_2 = \rho_3 = \bar{\rho}$, i.e. if the beam is equally stiff in the two bending directions (and thus in all directions), synchronous solutions of Eq. (44) exist; then $\sigma_2 = \sigma_3 = \bar{\sigma}$ and $\iota_2 = \iota_3 = \bar{\iota}$ and the circular frequency depends on the vibration mode as follows:

$$\bar{\sigma} = \left(\frac{\bar{\iota}\pi}{\bar{x}}\right)^2\sqrt{\frac{E\bar{\rho}^2}{\rho} + 3\left(\frac{\alpha_1\omega}{4}\right)^2}. \tag{45}$$

This shows that the circular frequency of any mode increases when $\omega \neq 0$ and suggests the possibility of damping oscillations through a fitting control (of bang–bang type) of ω (Brocato, 1996).

In Fig. 2, we show the graph of function (45) for the seven first modes.

When $\rho_2 \neq \rho_3$, solutions are still available but cumbersome. For instance, we have

$$\sigma_2 = \left(\frac{\iota_2\pi}{\bar{x}}\right)^2\sqrt{\frac{E\rho_2^2}{\rho} + \frac{\mathscr{A}^2(1+\mathscr{A})}{4}\left(\frac{\alpha_1\omega}{4}\right)^2}$$

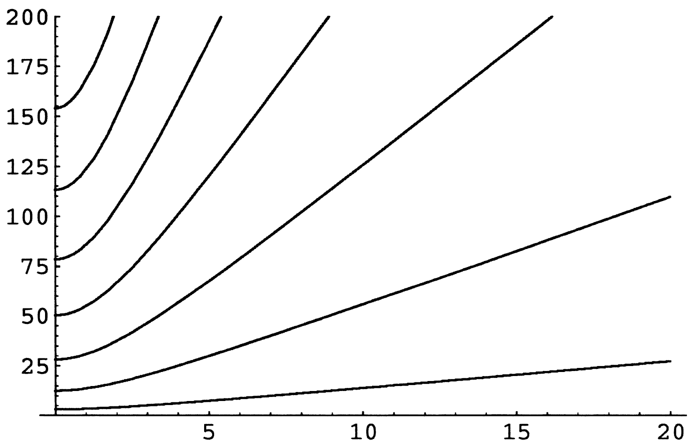

Fig. 2. Plot of $\bar{x}^2\sqrt{\rho/E\bar{\rho}}\bar{\sigma}$ versus $\sqrt{\rho/E\bar{\rho}}\alpha_1\omega$ for $\bar{\iota} = 1, \ldots, 7$.

with definitions:

$$\mathscr{A} := \frac{2\rho}{3E\rho_2^2}\left(\left(\frac{\alpha_1\omega}{4}\right)^2 + \frac{\mathscr{C}}{\mathscr{B}} + \mathscr{B}\right), \quad \mathscr{B} := \left(\mathscr{D} + \sqrt{-\mathscr{C}^3 + \mathscr{D}^2}\right)^{\frac{1}{3}},$$
$$\mathscr{C} = \left(\frac{\alpha_1\omega}{4}\right)^2\left(\left(\frac{\alpha_1\omega}{4}\right)^2 + 3\frac{E\rho_2^2}{\rho}\right), \qquad \mathscr{D} = \frac{27}{2}\frac{E^3\rho_2^4\rho_3^2}{\rho^3} - \left(\frac{\alpha_1\omega}{4}\right)^6 - \frac{9}{2}\left(\frac{\alpha_1\omega}{4}\right)^4\frac{E\rho_2^2}{\rho}.$$

6. Concluding remarks

We have proposed the mechanical model of a body endowed with a diffuse distribution of small gyroscopes. The system is reminiscent of a Cosserat type of continuum, but with peculiarities due to the nature of its fine details: we are imagining here a concrete structure, we are not aiming at a model for ether.

Thus, in Section 2.1, we have given details on how the linkage between a gyroscope and the main structure can be made.

The precise mathematical description of this linkage allows us to put on a firm base some ideas already presented in the literature (D'Eleuterio (1984) and D'Eleuterio and Hughes (1984)), and generalise them to a continuum model, the behaviour of which need not be linear and elastic.

In Section 4, after giving the equations of motion for the general system, we have investigated on the possibility of constrained evolutions of the microstructure. Especially Constraint 2, i.e. gyroscopes with axis fixed on the capsule and constant axial speed relative to the capsule itself, is interesting for applications in the field of controls.

In Section 5 we dealt with two examples, applying the model to a linear elastic homogeneous material respectively in the case of plane strain and in the case of Euler–Bernoulli beams. It appears then that the possibility of acting on the gyroscopes to influence – if not constrain – their evolution, allows us to control the gross motion of the system. We emphasise that not only diversion of energy to orthogonal movements is obtainable with this method, but also, through a sequence of fit bang–bang controls, damping of oscillations (Brocato, 1996).

Notice that, in concrete instances, optimal control can be achieved only through an appropriate arrangement of sensors, processors, actuators, which allows, through on-line fast processing, a response which can be taken to be instantaneous, having mechanical time constants in mind.

Acknowledgements

This research was carried out under the project "Modelli Matematici per la Scienze dei Material" of the Italian Ministry "MURST" and within the Computational Mechanics of Solids Network (COMES) of the Italian National Research Council.

References

Brocato, M., 1996. Bodies with controllable constitutive laws – some mathematical issues for active optimisation. In: Pierre, F.G., Jacques, T. (Eds.), Third Int. Conf. on Intelligent materials and Third European Conf. on Smart structures and materials. Proc. SPIE 2779, pp. 891–896.

Capriz, G., 1989. Continua with Microstructure. Springer, New York.

Capriz, G., Giovine, P., 1997. On microstructural inertia. Math. Mod. Meth. Appl. Sci. 7 (2), 211–216.

D'Eleuterio, G.M.T., 1984. On the theory of gyroelasticity. J. Appl. Mech. 55, 488–489.

D'Eleuterio, G.M.T., Hughes, P.C., 1984. Dynamics of gyroelastic continua. J. Appl. Mech. 51, 415–422.

Truesdell, C., Toupin, R., 1960. The Classical Field Theories. In: Flügge, S. (Ed.), Encyclopedia of physics, Vol. III/1. Principle of Classical Mechanics and Field Theory, Springer, Berlin.